On the Statistical Sufficiency of Matched Filter

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1 Introduction

In common text on communications and signal processing matched filters are derived from maximum output SNR of a linear time invariant system on a determined point. The question "is match filter best system (Linear or not) to detect a specific signal?" is not usually considered. Here we consider this question with restriction to Gaussian noise and framework of sufficient estimator.

2 Framework

Definition 1 *For $X = (X_1, ..., X_n) \sim f_X(x; \theta)$, an estimator t is sufficient if and only if $f(x_1, ..., x_n|t)$ is irelavant from θ[1]. In other words,*

$$Pr(x|t, \theta) = Pr(x|t) \tag{1}$$

Interpretation: A sufficient statistic is a statistic which has the property of sufficiency with respect to a statistical model and its associated unknown parameter, meaning that "no other statistic which can be calculated from the same sample provides any additional information as to the value of the parameter" [2].

Theorem 1 (Factorization Theorem) *For $X = (X_1, ..., X_n) \sim f_X(x; \theta)$, and $T = T(X_1, ..., X_n)$ is sufficient if and only if*

$$L(\theta) = \prod_i f(x_i; \theta) = g(t, \theta)h(X) \tag{2}$$

in which $t = T(x_1, ..., x_n)$ and $L(\theta)$ is the likelihood of θ.[proof:[3]]

3 Derivation

We know that liklihood function is defined as

$$L(\theta) = \prod_i f(x_i|\theta). \tag{3}$$

So,

$$l(\theta) = \ln(L(\theta)) = \sum_i \ln(f(x_i|\theta)). \tag{4}$$

For a signal recieved $x_i = s(t_i)$ and signal sent $m(t_i)$ and gaucian noise we have,

$$l(m(t)) = \ln(L(m(t))) = \sum_i \ln(f(s(t_i)|m(t_i))). \tag{5}$$

Generalizing this definition to continuous state we have,

$$l(m(t)) = \int \ln(f(s(t)|m(t)))dt. \tag{6}$$

On the other hand we know from Gaussian noise that for every t,

$$f(s(t)|m(t)) = ke^{\frac{(s(t)-m(t))^2}{2\sigma^2}}. \tag{7}$$

So,

$$\begin{aligned}
L(m(t)) =& exp\left(\int_0^T \ln(f(s(t)|m(t)))dt\right) && (8)\\
=& exp\left(\int_0^T \ln(ke^{\frac{(s(t)-m(t))^2}{2\sigma^2}})dt\right) && (9)\\
=& exp\left(\int_0^T \ln(k) + \frac{(s(t)-m(t))^2}{2\sigma^2}dt\right) && (10)\\
=& exp\left(T\ln(k) + \int_0^T \frac{(s(t)-m(t))^2}{2\sigma^2}dt\right) && (11)\\
=& exp\left(T\ln(k) + \int_0^T \frac{s^2(t)+m^2(t)+2s(t)m(t)}{2\sigma^2}dt\right) && (12)\\
=& exp(Tln(k))exp\left(\frac{1}{2\sigma^2}\left(P_s + P_m + 2\int_0^T m(t)s(t)dt\right)\right) && (13)\\
=& \underbrace{exp(Tlnk)exp(\frac{1}{2\sigma^2}P_s)}_{f(s(t))}\underbrace{exp(\frac{1}{2\sigma^2}P_m)exp\left(\frac{1}{2\sigma^2}\int_0^T s(t)m(t)\right)}_{g(t,m(t))}. && (14)
\end{aligned}$$

So we have that $t = \int_0^T s(t)m(t)dt$ is a sufficient estimator. ■

References

[1] E. Ziegel, "Statistical inference," *Technometrics*, vol. 44, no. 4, pp. 407–408, 2002.

[2] R. Fisher, "On the mathematical foundations of theoretical statistics," *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, vol. 222, no. 594-604, p. 309, 1922.

[3] R. Hogg and A. Craig, "Introduction to mathematical statistics, 1995."