

Linear Control of Flexible Joint

Final Report for “Linear Control Laboratory”
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1 Modelling a flexible joint

Regarding the guidelines in lab sheet, we consider the fundamental equations of system as following:

$$\begin{cases} J_{eq}\ddot{\theta} + J_{arm}(\ddot{\alpha} + \ddot{\theta}) = T_{output} - B_{eq}\dot{\theta} \\ J_{arm}(\ddot{\theta} + \ddot{\alpha}) + K_{stiff}\alpha = 0 \end{cases} \quad (1)$$

The motor model is assumed as following:

$$T_{output} = \frac{\eta_m \eta_g K_t K_g (V_m - K_g K_m \dot{\theta})}{R_m} \quad (2)$$

In which constants are shown in Table.1.

Constant	Value
K_{stiff}	1.2485
B_{eq}	0.0040
J_{arm}	0.0035
J_{eq}	0.0026

Table 1. The constant values of system's dynamics

The motor model is assumed as shown in Table.2.

Constant	Value
R_m	2.6
K_g	70
$K_t = K_m$	0.00767
η_g	0.9
η_m	0.69

Table 2. The constant values of the motor model

2 State-space model

In this section we will derive the state-space model of the system.

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{A}\mathbf{X} + \mathbf{B}\mathbf{U} \\ \mathbf{Y} = \mathbf{C}\mathbf{X} + \mathbf{D}\mathbf{U} \end{cases}$$

$$\mathbf{X} = \begin{bmatrix} \alpha \\ \dot{\alpha} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

By rearranging the fundamental equations, we can easily achieve to following state equations:

$$\begin{bmatrix} \dot{\alpha} \\ \ddot{\alpha} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_{stiff}(J_{eq}+J_{arm})}{J_{arm}J_{eq}} & 0 & 0 & -\frac{\eta_m\eta_g K_t K_m K_g^2}{R_m J_{eq}} - \frac{B_{eq}}{J_{eq}} \\ 0 & 0 & 0 & 1 \\ \frac{K_{stiff}}{J_{eq}} & 0 & 0 & +\frac{\eta_m\eta_g K_t K_g}{R_m J_{eq}} + \frac{B_{eq}}{J_{eq}} \end{bmatrix} \begin{bmatrix} \alpha \\ \dot{\alpha} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\eta_m\eta_g K_t K_g}{R_m J_{eq}} \\ 0 \\ -\frac{\eta_m\eta_g K_t K_g}{R_m J_{eq}} \end{bmatrix} V_m$$

Thus the state matrices become as what follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -836.9 & 0 & 1 & -28.0193 \\ 0 & 0 & 0 & 1 \\ 480.19 & 0 & 0 & 50.8 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 \\ 49.3 \\ 0 \\ -49.3 \end{bmatrix}$$

By defining the output to be $\mathbf{Y} = \begin{bmatrix} \alpha \\ \theta \end{bmatrix}$ we can find the output state-space equations as following:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\mathbf{D} = [0]$$

3 Transfer functions

In addition to state-space model, here we derive the transfer functions of the system parameters.

$$(J_{eq} + J_{arm}) s^2 \theta + J_{arm} s^2 \alpha + \left(\frac{\eta_m \eta_g K_t K_m K_g^2}{R_m} + B_{eq} \right) s \theta = \frac{\eta_m \eta_g K_t K_g}{R_m} V_m \quad (3)$$

$$J_{arm} s^2 \theta + J_{arm} s^2 \alpha + K_{stiff} \alpha = 0 \quad (4)$$

By combining equations (3) and (4), we can find:

$$\alpha = \frac{-J_{arm} s^2}{J_{arm} s^2 + K_{stiff}} \theta \quad (5)$$

$$\frac{\theta}{V_m} = \frac{(J_{arm} s^2 + K_{stiff}) \frac{\eta_m \eta_g K_t K_g}{R_m}}{J_{arm} J_{eq} s^4 + J_{arm} \left(\frac{\eta_m \eta_g K_t K_m K_g^2}{R_m} + B_{eq} \right) s^3 + K_{stiff} (J_{eq} + J_{arm}) s^2 + K_{stiff} \left(\frac{\eta_m \eta_g K_t K_m K_g^2}{R_m} + B_{eq} \right) s}$$

$$\frac{\alpha}{V_m} = \frac{-J_{arm} s^2 \frac{\eta_m \eta_g K_t K_g}{R_m}}{J_{arm} J_{eq} s^4 + J_{arm} \left(\frac{\eta_m \eta_g K_t K_m K_g^2}{R_m} + B_{eq} \right) s^3 + K_{stiff} (J_{eq} + J_{arm}) s^2 + K_{stiff} \left(\frac{\eta_m \eta_g K_t K_m K_g^2}{R_m} + B_{eq} \right) s}$$

By substituting values of the parameters, we have:

$$\frac{\theta}{V_m} = \frac{49.32s^2 + 1.759 \times 10^4}{s^4 + 28.02s^3 + 836.9s^2 + 9995s} \quad (6)$$

$$\frac{\alpha}{V_m} = \frac{-49.32s^2}{s^4 + 28.02s^3 + 836.9s^2 + 9995s} \tag{7}$$

4 Controller design and simulation

As it is demanded, the controller show afford the values in Table.3.

Parameter	Desired Value
Settling time	less than 0.7sec
Overshoot	less than 5%
Maximum deviation of α	9°

Table 3. Desired values for the controller

Simplifying the desired point, we can write:

$$T_s < 0.7sec \Rightarrow \frac{4}{\xi\omega_n} < 0.7 \Rightarrow \xi\omega_n > 5.72$$

$$e^{\frac{\xi\omega_n}{\sqrt{1-\xi^2}}} < 0.05 \Rightarrow \xi > 0.6901$$

By the of the MATLAB's SISO tool we design the following controllers:

Controller	$C(s) = 2.1 \frac{s+6}{s+5}$	$C(s) = 2.1 \frac{s+7}{s+6}$	$C(s) = 2.1 \frac{s+8}{s+7}$	$C(s) = 2 \frac{s+10}{s+8}$	$C(s) = 2 \frac{s+11}{s+9}$	$C(s) = 2 \frac{s+12}{s+10}$
T_s (sec)	0.49	0.49	0.49	0.49	0.49	0.49
$O.V.$ (%)	4.1	3.2	2.5	3.9	3.1	2.5
α_{max} (deg)	6.2	6.2	6.2	6.1	6.1	6.1

The controlling schematic of the system is shown in Fig.1.

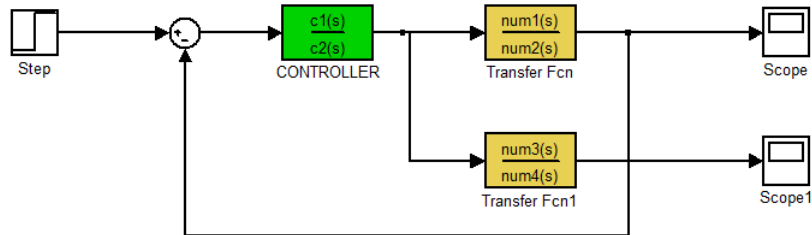


Fig. 1. The controlling system schematic.

A sample simulation of the system's response to unit pulse input is shown in Fig.2.

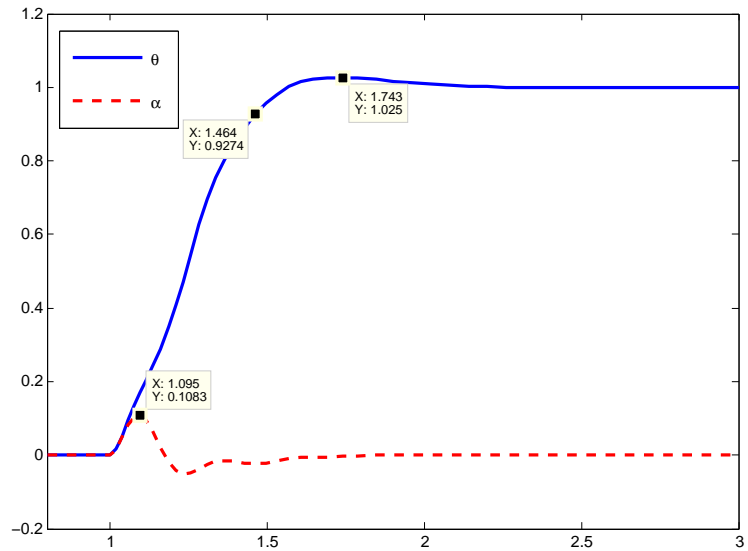


Fig. 2. Unit pulse response.